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Direct Computation of Lower-Bound Dynamic Buckling Loads of Imperfection-Sensitive Systems

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The computation of lower-bound dynamic buckling loads of imperfection-sensitive systems is examined under step load of infinite duration. An extended system based on the energy criterion for establishing the lower-bound dynamic buckling loads without solving the highly nonlinear initial-value problems is proposed. The newly introduced scaling parameters are nonsingular solutions to the extended system; thus, standard methods can be used to compute them. Using the extended system, one can directly obtain the lower-bound dynamic buckling loads without tracing the postbuckling equilibrium paths. An efficient implementation of Newton's method for solving the extended system is presented, and numerical examples are given.

I. Introduction

FOR a discrete nonlinear autonomous structural system under step load of infinite duration, one usually seeks the exact value λ_d of dynamic buckling load by integrating the equations of motion subject to given initial conditions. In some cases, the numerical integrations are quite time-consuming. This is particularly true when an unbounded motion is initiated after a very long period of time or, in case of the existence of chaotic-like phenomena, because of sensitivity to initial conditions or because of damping.[1–8] Therefore, it is highly desirable to supplement numerical simulation by other solutions, such as approximate or lower- or upper-bound estimates for the dynamic buckling load. Such estimates not only help us to monitor the accuracy of the numerical algorithms but also provide fast and inexpensive approximations that can be very useful for structural design purposes.[1–8]

Without solving the highly nonlinear system of ordinary differential equations, the lower- or upper-bound buckling estimates based on energy criteria for initially imperfect discrete as well as continuous structural systems, under step load of infinite duration, have been established by Kounadis and associates[1–8] and Simitses.[9] Let λ be the load parameter and ε a measure of the magnitude of a given imperfection pattern. We assume that the imperfect system under the same loading λ applied statically exhibits a limit point instability. Then, the limit point load (static buckling load) λ_s is an upper bound of the dynamic buckling load λ_d. Determination of λ_s has been discussed by numerous investigators (see, e.g., Ref. 10 and references cited therein). The lower-bound dynamic buckling load $\tilde{\lambda}_d$ and the corresponding displacement u_d are associated with an equilibrium point of the unstable postbuckling path, where the total potential energy is equal to zero. The lower-bound estimate is quite close to the exact dynamic buckling load, as shown in Refs. 1–8. The upper and lower bounds of dynamic buckling loads are different for the same magnitude of imperfections, as shown in the asymptotic analyses in Refs. 10 and 11.

Based on the energy criterion for establishing the lower-bound dynamic buckling load, three approaches can be used for its numerical realization. The first approach[9] is to solve the corresponding static equilibrium problem and compute the value of the total potential energy at every static unstable equilibrium point. The load value, for which the value of the total potential energy at this unstable point becomes zero, is a measure of the lower-bound dynamic buckling load. This method, however, requires a separate analysis for each magnitude of a given imperfection pattern to be considered. Also, equilibrium paths may have strong curvatures in the vicinity of bifurcation points, rendering them difficult to track numerically. The second possible procedure is finding the simultaneous solution of the equilibrium equation along with the potential energy being equal to zero for a given imperfection, subject to the condition that the static equilibrium position obtained from the solution is unstable. The approach needs special attention because it easily can yield physically unacceptable solutions that arise from the nonlinearity of the problem.[9] The third approach[11,12] is to find the lower-bound dynamic buckling load by perturbing the simultaneous equations resulting from the energy criterion. This approach, however, involves complex terms, e.g., third- and fourth-order derivatives of potential energy. In addition, the range of validity of this method is restricted because such results often are based on the lower-order asymptotic analyses.

The goal of this paper is to present an alternative method for determining the lower-bound dynamic buckling loads that can overcome the previous reservations. The new method will be constructed by introducing an extended system (see, e.g., Refs. 13 and 14). By introducing some new scaling parameters, which have their origin in the Lyapunov–Schmidt–Koiter approach[15,16] and the perturbation expansion technique,[11] we bridge the gap between the bifurcation point of the perfect system and the zero potential energy point of the postbuckling path of the imperfect system by use of the extended system. The solution to the extended system can be expressed as smooth functions of ξ, the amplitude of projection of displacement on the normalized buckling mode of the perfect system. By analyzing the stability of the obtained solutions,[11] we can choose the direction of the continuation parameter ξ so that the instability condition is satisfied automatically. Finally, without tracking the equilibrium paths of the imperfect system, we can directly obtain the lower-bound dynamic buckling loads by using a continuation of ξ from $\xi = 0$ with the extended system. The implementation of Newton's method for solving the extended system—a partitioning procedure—also is discussed. Two examples are used to illustrate the present method.

II. Basic Formulas

The term *imperfect system* is used repeatedly and denotes a more detailed structural model, which simulates some or all of the unintended deviations of the real system from the perfect model. These unintended deviations are denoted collectively as imperfections. It is understood that the imperfections have been normalized so that, for zero magnitude of the imperfection patterns, the imperfect system is reduced to the perfect one.

Consider a discrete autonomous imperfect system. Let the potential energy of the imperfect system be given by $V(u;\lambda;w)$, where u denotes the additional displacement of the imperfect system from its initial configuration, λ the loading parameter, and w the imperfection of the system. The loading λ is considered to be the main control parameter for the occurrence of static and dynamic bifurcations. Furthermore, let U and W denote, respectively, spaces of displacement and imperfection of the system. We introduce two inner

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products in the two spaces denoted by $(u_1; u_2)$ for $u_i \in U\,(i = 1; 2)$ and $[\mathrm{w}_1; \mathrm{w}_2]$ for $\mathrm{w}_i \in W\,(i = 1; 2)$, and the corresponding norms are $\|u\| = (u; u)^{1/2}$ for $u \in U$ and $|\mathrm{w}| = [\mathrm{w}; \mathrm{w}]^{1/2}$ for $\mathrm{w} \in W$, respectively. It is convenient to distinguish between the imperfection magnitude $\varepsilon = |\mathrm{w}|$ and the normalized imperfection pattern $\bar{u} = \mathrm{w}/|\mathrm{w}|$. Thus, the potential energy V may be written as $V(u; \lambda; \varepsilon\bar{u})$. Furthermore, for all λ and ε the potential energy V can always be chosen to obey the condition

$$V(0; \lambda; \varepsilon\bar{u}) = 0 \tag{1}$$

Throughout, we use the following notation: various functional (Fréchet) derivatives of the potential energy with respect to u and w are denoted by subscripts $(\,)_u$ and $(\,)_{\mathrm{w}}$, respectively, and the ordinary partial derivatives with respect to λ are denoted by a subscript $(\,)_\lambda$, etc.

By Eq. (1), the potential energy of the imperfect system, under step load of infinite duration, is zero at the initial moment. Consequently, according to the energy criterion,[1–9] the lower-bound dynamic buckling load can be obtained by requiring that the potential energy V at a certain unstable static equilibrium position be equal to zero. That is, for each magnitude ε of a given imperfection pattern $\bar{u}$, the lower-bound dynamic buckling load $\tilde{\lambda}_d$ and the corresponding displacement u_d are the simultaneous solutions (for unknowns u and λ) of the nonlinear equilibrium equations

$$V_u(u; \lambda; \varepsilon\bar{u}) = 0 \tag{2}$$

along with

$$V(u; \lambda; \varepsilon\bar{u}) = 0 \tag{3}$$

subject to the condition that the static equilibrium position determined by the solution of Eqs. (2) and (3) is unstable.

As pointed out in Sec. I, it can easily yield physically unacceptable solutions to directly solve system of Eqs. (2) and (3) with respect to u and λ for a given value of ε. We will propose a new method to attack this problem in next section. To this end, the following conditions are presented:

We assume that, for the perfect system, there exists a trivial major equilibrium solution $u = u_0(\lambda) = 0$ as the load increases from zero, i.e.,

$$V_u(0; \lambda; 0) \equiv 0 \tag{4}$$

Let $\lambda = \lambda_c$ be the buckling load for the perfect system, assumed to be simple, with corresponding buckling mode u_1 normalized by $\|u_1\| = 1$. In mathematical terms,

$$V^c_{uu} u_1 = 0 \tag{5}$$

where superscript c denotes the corresponding derivatives of potential energy function V calculated at $(u; \lambda; \mathrm{w}) = (0; \lambda_c; 0)$. We further assume that, when load parameter λ passes increasingly through its critical value λ_c, the trivial major equilibrium solution becomes unstable from stable[17–20]:

$$V^c_{uu\lambda} u_1^2 < 0 \tag{6}$$

We also assume that the effect of imperfections is of the first order[17–20]:

$$V^c_{u\mathrm{w}} u_1 \bar{u} \neq 0 \tag{7}$$

III. Extended System for Computing the Lower-Bound Dynamic Buckling Loads

For the stable solution of a nonlinear system, using a Newton-like method, for example, it is important that the solution be nonsingular.[13] Based on the Lyapunov–Schmidt–Koiter approach[15,16] and the perturbation expansion technique,[11] the following extended system for determining the lower-bound dynamic buckling loads of discrete autonomous imperfect systems under step load of infinite duration is presented:

$$E(\mathrm{v}; 3; \tau; \square) := \begin{Bmatrix} g_1(\mathrm{v}; 3; \tau; \square) \\ g_2(\mathrm{v}; 3; \tau; \square) \\ g_3(\mathrm{v}; 3; \tau; \square) \end{Bmatrix} = 0 \tag{8a}$$

where

$$g_1 := \begin{cases} \dfrac{V_u\left(\square u_1 + \square^2\mathrm{v};\ \lambda_c + \square 3;\ \square^2\tau\bar{u}\right)}{\square^2}; & \text{if } \square \neq 0 \\ V^c_{uu}\mathrm{v} + 3 V^c_{uu\lambda} u_1 + \tau V^c_{u\mathrm{w}}\bar{u} + \dfrac{1}{2} V^c_{uuu} u_1^2; & \text{if } \square = 0 \end{cases} \tag{8b}$$

$$g_2 := \begin{cases} \dfrac{V\left(\square u_1 + \square^2\mathrm{v};\ \lambda_c + \square 3;\ \square^2\tau\bar{u}\right)}{\square^3}; & \text{if } \square \neq 0 \\ \dfrac{1}{2} 3 V^c_{uu\lambda} u_1^2 + \tau V^c_{u\mathrm{w}} u_1\bar{u} + \dfrac{1}{6} V^c_{uuu} u_1^3; & \text{if } \square = 0 \end{cases} \tag{8c}$$

$$g_3 := (u_1; \mathrm{v}) \tag{8d}$$

and

$$E : R^n \times R \times R \times R \to R^n \times R \times R \tag{8e}$$

Using the Taylor expansion, one can derive that g_1 and g_2 are continuous.

Consider the following system of equations:

$$g_1(\mathrm{v}; 3; \tau; 0) = 0; \qquad g_2(\mathrm{v}; 3; \tau; 0) = 0$$
$$g_3(\mathrm{v}; 3; \tau; 0) = 0 \tag{9}$$

It has the solution

$$(\mathrm{v}; 3; \tau) = (\mathrm{v}_0; 3_0; \tau_0)$$

where

$$\mathrm{v}_0 = \mathrm{v}_{00} - (u_1; \mathrm{v}_{00})u_1; \qquad 3_0 = -\frac{2V^c_{uuu}u_1^3}{3V^c_{uu\lambda}u_1^2}$$
$$\tau_0 = \frac{V^c_{uuu}u_1^3}{6V^c_{u\mathrm{w}}u_1\bar{u}} \tag{10}$$

and v_{00} is a particular solution to the following equation:

$$V^c_{uu}\mathrm{v} + 3_0 V^c_{uu\lambda} u_1 + \tau_0 V^c_{u\mathrm{w}}\bar{u} + \tfrac{1}{2} V^c_{uuu} u_1^2 = 0 \tag{11}$$

Based on conditions (1), (4), and (5), the homogeneous system associated with the linearization of $E = 0$ in Eq. (8) at $\square = 0$ with respect to $(\mathrm{v}; 3; \tau)$ has the form

$$V^c_{uu}\delta\mathrm{v} + \delta 3\, V^c_{uu\lambda} u_1 + \delta\tau\, V^c_{u\mathrm{w}}\bar{u} = 0 \tag{12a}$$

$$\tfrac{1}{2}\delta 3\, V^c_{uu\lambda} u_1^2 + \delta\tau\, V^c_{u\mathrm{w}} u_1\bar{u} = 0 \tag{12b}$$

$$(u_1; \delta\mathrm{v}) = 0 \tag{12c}$$

Taking inner product with u_1 in the two sides of Eq. (12a), using Eq. (5) one achieves

$$\delta 3\, V^c_{uu\lambda} u_1^2 + \delta\tau\, V^c_{u\mathrm{w}} u_1\bar{u} = 0 \tag{13}$$

Under conditions (6) and (7), the determinant of coefficient matrix of Eqs. (12b) and (13) is nonzero; thus we deduce $\delta 3 = \delta\tau = 0$.

Substituting $\delta 3 = \delta\tau = 0$ into Eq. (12a) gives

$$V^c_{uu}\delta\mathrm{v} = 0 \tag{14}$$

From Eq. (5), the solution to Eq. (14) is given as

$$\delta\mathrm{v} = c u_1 \tag{15}$$

Use of Eqs. (15) and (12c) leads to $c = 0$; hence

$$\delta\mathrm{v} = 0$$

Then we have the following theorem:

Theorem. Let potential energy function $V(u; \lambda; \varepsilon\bar{u})$ satisfy conditions (1) and (4–7). Then, for $\square = 0$, extended system (8) has a solution $(\mathrm{v}_0; 3_0; \tau_0)$ and its linearization with respect to $(\mathrm{v}; 3; \tau)$, at this solution, is nonsingular.

We draw some important conclusions from the nonsingularity of the extended system $E = 0$ in Eq. (8) at $\square = 0$. Based on the implicit-function theorem applied to ($|\Box| < \Box_0$) $E.\mathsf{v}; 3; ¿; \Box/ = 0$, there exists a locally unique smooth solution curve $.\mathsf{v}.\Box/; 3.\Box/; ¿.\Box//$ of $E = 0$ passing through $.\mathsf{v}_0; 3_0; ¿_0/$ at $\Box = 0$.

On the other hand, for $\Box \neq 0$, from Eqs. (8a–d), one derives

$$V_u\big(\Box u_1 + \Box^2 \mathsf{v}.\Box/; \,{}_{¸c} + \Box\, 3.\Box/; \Box^2 ¿.\Box/\bar{u}\big) = 0$$
$$V\big(\Box u_1 + \Box^2 \mathsf{v}.\Box/; \,{}_{¸c} + \Box\, 3.\Box/; \Box^2 ¿.\Box/\bar{u}\big) = 0 \qquad (16)$$

Hence, for the given imperfection pattern $\bar{u}$, $.u; {}_¸/ = .\Box u_1 + \Box^2\mathsf{v}.\Box/, {}_{¸c} + \Box\, 3.\Box//$ corresponds to a solution to the system of Eqs. (2) and (3) for the imperfection magnitude $" = \Box^2 ¿.\Box/$. Using the conclusion of stability analysis in Wu,[11] we see that, only if $\Box\, 3.\Box/ < 0$, $\tilde{¸}_d := {}_{¸c} + \Box\, 3.\Box/$ is the lower-bound dynamic buckling load for the imperfection magnitude $" = \Box^2 ¿.\Box/$ and the corresponding displacement $u_d = \Box u_1 + \Box^2 \mathsf{v}.\Box/$.

Summarizing these results, we have the following corollary:

Corollary. Let the conditions of the theorem be satisfied. Then, there exists a locally smooth solution curve $.\mathsf{v}.\Box/, 3.\Box/, ¿.\Box// \in R^n \times R \times R$ of $E.\mathsf{v}; 3; ¿; \Box/ = 0$ such that $\mathsf{v}.0/ = \mathsf{v}_0$, $3.0/ = 3_0$ and $¿.0/ = ¿_0$. Furthermore, for $\Box \neq 0$, ${}_{¸c} + \Box\, 3.\Box/$ is the lower-bound dynamic buckling load with respect to imperfection magnitude $\Box^2 ¿.\Box/$ for given imperfection pattern $\bar{u}$ when $\Box\, 3.\Box/ < 0$.

Now we can propose a procedure for the computation of lower-bound dynamic buckling loads for imperfect systems.

1) Determine the buckling load ${}_¸ = {}_{¸c}$ and corresponding buckling mode u_1 of the perfect system by solving an eigenvalue problem.

2) For a given imperfection pattern $\bar{u}$, extended system (8) (i.e., $E = 0$) then can be used to compute the lower-bound dynamic buckling loads ${}_{¸c} + \Box\, 3.\Box/$ and the corresponding displacement $\Box u_1 + \Box^2 \mathsf{v}.\Box/$ for the imperfection magnitude $\Box^2 ¿.\Box/$ by using a continuation of $\Box$ from $\Box = 0$ along the direction of $\Box$ satisfying $\Box\, 3.\Box/ < 0$.

Remark. The nonsingularity of $.\mathsf{v}; 3; ¿/ = .\mathsf{v}_0; 3_0; ¿_0/$ as the solution to extended system (8) with $\Box = 0$ guarantees that we can succeed in the numerical continuation in step (2).

Extended system (8) can be solved by Newton's method. In the practical realization, one can take advantage of a partitioning technique.[21] For given $\Box \neq 0$, we have to solve the system $E.z; \Box/ = 0$ by the iterations

$$E_z.z^k; \Box/\pm z^k = -E.z^k; \Box/; \qquad z^{k+1} = z^k + \pm z^k; \qquad k = 1; 2; \ldots \qquad (17)$$

where $z^k = .\mathsf{v}^k; 3^k; ¿^k/$ and $\pm z^k = .\pm\mathsf{v}^k; \pm 3^k; \pm ¿^k/$. Equation (17) can be written as [$V_u^k := V_u.\Box u_1 + \Box^2\mathsf{v}^k; {}_{¸c} + \Box\, 3^k; \Box^2 ¿^k \bar{u}/$ etc.]

$$A^k \pm\mathsf{v}^k + b^k \pm 3^k + c^k \pm ¿^k = r^k \qquad (18a)$$

$$p^k \pm\mathsf{v}^k + q^k \pm 3^k + y^k \pm ¿^k = s^k \qquad (18b)$$

$$.u_1; \pm\mathsf{v}^k/ = t^k \qquad (18c)$$

where

$$A^k = V_{uu}^k; \qquad b^k = \frac{V_{u¸}^k}{\Box}; \qquad c^k = V_{u\mathsf{w}}^k \bar{u}$$

$$p^k = \frac{V_u^k}{\Box}; \qquad q^k = \frac{V_¸^k}{\Box^2}; \qquad y^k = \frac{V_{\mathsf{w}}^k \bar{u}}{\Box}$$

$$r^k = -\frac{V_u^k}{\Box^2}; \qquad s^k = -\frac{V^k}{\Box^3}; \qquad t^k = -\big(u_1; \mathsf{v}^k\big)$$

Direct solution of systems (18a–c) with Gaussian elimination might require full pivoting strategy to avoid severe accumulation of round-off errors. However, full pivoting destroys the bordered structure of the coefficient matrix. We solve system (18a–c) by means of a partition of the coefficient matrix.

For $\Box = 0$, the tangent stiffness matrix $K^c = V_{uu}^c$ at the bifurcation point is of rank $(n-1)$ and cannot be inverted. Let j be the largest component in the buckling mode u_1. The submatrix $\bar{K}^c$ achieved by deleting the jth row and the jth column from K^c is nonsingular. Hence, if $.\mathsf{v}^k; 3^k; ¿^k; \Box/$ is close to $.\mathsf{v}_0; 3_0; ¿_0; 0/$, the $.n-1/ \times .n-1/$ submatrix B^k of A^k that is obtained by deleting the jth row and the jth column from matrix $A^k .= .a_{ij}^k/_{n\times n}/$ will be nonsingular.

Equation (18a) can be split into the following two groups:

$$B^k \pm\bar{\mathsf{v}}^k = -\bar{b}^k \pm 3^k - \bar{c}^k \pm ¿^k - \bar{d}^k \pm\mathsf{v}_j^k + \bar{r}^k \qquad (19a)$$

$$\sum_{l=1}^{n} a_{jl}^k \pm\mathsf{v}_l^k + b_j^k \pm 3^k + c_j^k \pm ¿^k = r_j^k \qquad (19b)$$

where $\pm\bar{\mathsf{v}}^k; \bar{b}^k; \bar{c}^k; \bar{d}^k$, and $\bar{r}^k$ are $n-1$ dimensional vectors defined by

$$\pm\bar{\mathsf{v}}^k = \big(\pm\mathsf{v}_1^k; \pm\mathsf{v}_2^k; \ldots; \pm\mathsf{v}_{j-1}^k; \pm\mathsf{v}_{j+1}^k; \ldots; \pm\mathsf{v}_{n-1}^k; \pm\mathsf{v}_n^k\big)^t$$

$$\bar{b}^k = \big(b_1^k; b_2^k; \ldots; b_{j-1}^k; b_{j+1}^k; \ldots; b_{n-1}^k; b_n^k\big)^t$$

$$\bar{c}^k = \big(c_1^k; c_2^k; \ldots; c_{j-1}^k; c_{j+1}^k; \ldots; c_{n-1}^k; c_n^k\big)^t$$

$$\bar{d}^k = \big(a_{1j}^k; a_{2j}^k; \ldots; a_{j-1;j}^k; a_{j+1;j}^k; \ldots; a_{n-1;j}^k; a_{nj}^k\big)^t$$

$$\bar{r}^k = \big(r_1^k; r_2^k; \ldots; r_{j-1}^k; r_{j+1}^k; \ldots; r_{n-1}^k; r_n^k\big)^t$$

We can get the solution to Eq. (19a) in terms of $\pm 3^k; \pm ¿^k$, and $\pm\mathsf{v}_j^k$, i.e.,

$$\pm\bar{\mathsf{v}}^k = h_1 + \pm 3^k h_2 + \pm ¿^k h_3 + \pm\mathsf{v}_j^k h_4 \qquad (20a)$$

by solving the systems

$$B^k h_1 = \bar{r}^k; \qquad B^k h_2 = -\bar{b}^k$$
$$B^k h_3 = -\bar{c}^k; \qquad B^k h_4 = -\bar{d}^k \qquad (20b)$$

Substitutions of Eq. (20a) into Eqs. (18b), (18c), and (19b) lead to a linear system of three equations for three unknown $\pm 3^k; \pm ¿^k; \pm\mathsf{v}_j^k$, and its solution can be obtained easily. Finally, from Eq. (20a), one can get $\pm\bar{\mathsf{v}}^k$ and then $\pm\mathsf{v}^k$.

In solving extended system (8), we never use $\Box = 0$, so that, when applying Newton's method, derivatives of the tangent stiffness matrix need not be computed. Furthermore, in the first continuation step starting from $\Box = 0$, we may choose $.\mathsf{v}^0; 3^0; ¿^0/ = .0; 0; 0/$. After that, one can use predictor-solver methods to determine functions $\mathsf{v}.\Box/; 3.\Box/$, and $¿.\Box/$.

IV. Numerical Examples

Example 1. Ziegler's two-degree-of-freedom cantilevered model under step load of infinite duration.

The model shown in Fig. 1 consists of two rigid weightless links of equal length l, interconnected with each other and being supported by frictionless hinges and corresponding nonlinearly elastic rotational springs of quadratic type. The unstressed configuration is specified by the initial geometric imperfections $"_1$ and $"_2$. The model at its top end is subjected to a step load P of infinite duration.

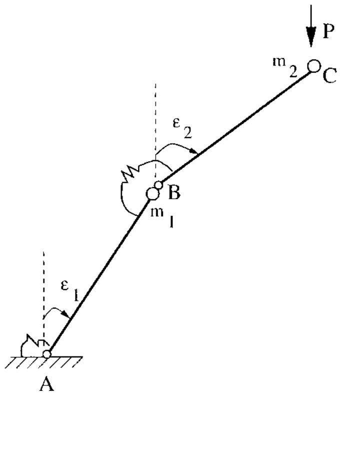

Fig. 1 Ziegler's cantilevered model under step load of infinite duration.

The total potential energy of this structural system is given by

$$V.µ_1; µ_2; ¸; "_1; "_2/ = \frac{1}{2}µ_1^2 + \frac{1}{3}±_1µ_1^3 + \frac{1}{2}.µ_2 - µ_1/^2$$
$$+ \frac{1}{3}±_2.µ_2 - µ_1/^3 - ¸[\cos "_1 - \cos .µ_1 + "_1/$$
$$+ \cos "_2 - \cos .µ_2 + "_2/] \qquad (21)$$

where $µ_1$ and $µ_2$ are the incremental angles of deformation of the system $(u := .µ_1; µ_2/^t)$, k is the linear spring component common for both springs, and $±_i .i = 1; 2/$ are the nonlinear components of the corresponding quadratic springs; $±_i > 0\ . < 0/$ express that the corresponding spring is of hard (soft) type, $¸ := Pl=k$. Thus, the equilibrium equation of the system is as follows:

$$µ_1 + ±_1µ_1^2 - µ_2 + µ_1 - ±_2.µ_2 - µ_1/^2 - ¸ \sin.µ_1 + "_1/ = 0$$
$$µ_2 - µ_1 + ±_2.µ_2 - µ_1/^2 - ¸ \sin.µ_2 + "_2/ = 0 \qquad (22)$$

Let the imperfection

$$w = ."_1; "_2/^t = "\bar{u}; \qquad \bar{u} := .a; b/^t; \qquad a^2 + b^2 = 1 \qquad (23)$$

In the following numerical computations, we take $±_1 = -2{:}5$, $±_2 = -0{:}75$. At the first bifurcation point of the perfect model, the buckling load $¸_c = 0{:}381966$ and the corresponding buckling mode $u_1 = .0{:}525731; 0{:}850651/^t$. We use extended system (8) to determine v.□/; 3.□/; ¿.□/ by the methods described in Sec. III. Figure 2 shows the variation of the lower-bound dynamic buckling load $\tilde{¸}_d$ with the imperfection magnitude " for three different imperfection patterns $.a; b/^t = .-0{:}71; 0{:}71/^t$, $.1{:}0; 0{:}0/^t$, and $.0{:}53; 0{:}85/^t$. For comparison, we also plot the corresponding asymptotic lower-bound dynamic buckling loads [up to $O."^{3=2}/$] derived by using the method given in Ref. 11.

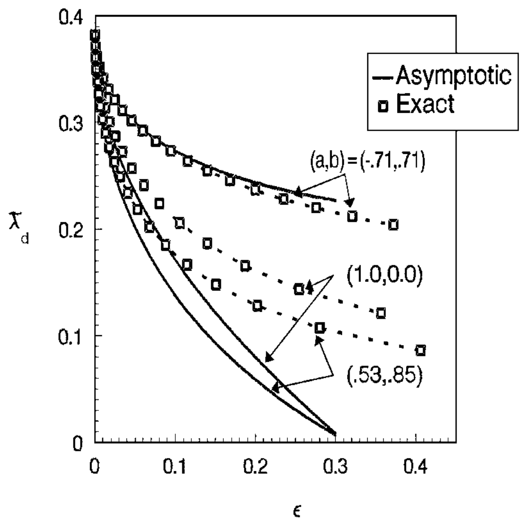

Fig. 2 Variation of lower-bound dynamic buckling load with imperfection magnitude for three different imperfection patterns.

From Fig. 2, one observes that, as imperfection magnitude increases, the lower-bound dynamic buckling load decreases; and along with $\bar{u} = .0{:}53; 0{:}85/^t$, which has the pattern of the buckling mode, the lower-bound dynamic load decreases most rapidly. Another important fact is that the asymptotic analysis generally is a reasonable approximation only for a small range of the imperfection magnitude.

Example 2. A simply supported beam on a nonlinear softening elastic foundation subjected to axial step load of infinite duration (Fig. 3).

The nondimensional potential energy of the beam is[15]

$$8.u; ¸; w/ = \int_0^l \left(\frac{1}{2}u''^2 - \frac{1}{2}¸u'^2 + \frac{1}{2}u^2 - \frac{1}{4}u^4 - ¸w'u'\right) dx \qquad (24a)$$

The nondimensional axial coordinate x, incremental lateral displacement u, axial load $¸$, and initial geometric imperfection w are related to the corresponding physical quantities by

$$x = \left(\frac{k_1}{EI}\right)^{\frac{1}{4}} X; \qquad u = \left(\frac{k_3}{k_1}\right)^{\frac{1}{2}} W;$$
$$¸ = \frac{P}{.EIk_1/^{\frac{1}{2}}}; \qquad w = \left(\frac{k_3}{k_1}\right)^{\frac{1}{2}} \bar{W} \qquad (24b)$$

The incremental lateral deflection W of the beam is restrained by a continuous elastic foundation that produces a nonlinear restraining force per unit length of $k_1W - k_3W^3$, with $k_1 > 0; k_3 > 0$. For the sake of illustration, we take $l \equiv .k_1=EI/^{1=4}L = ¼$ in the following discussion.

We choose a uniform mesh size, $h = ¼=.N + 1/$, N: integer, and nodes $x_i = ih$, $i = 1; 2; \ldots; N$. Starting from Eq. (24) and using difference-variation (evaluate the integral with trapezoid formula and use central difference quotient approximations to u'' and u') discretization to it leads to the corresponding finite-dimensional potential energy function V:

$$V.u; ¸; w/ = \frac{1}{2}h\sum_1^N\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right)^2$$
$$-\frac{1}{2}h¸\left[\frac{1}{2}\frac{u_1^2}{h^2} + \sum_1^N\left(\frac{u_{i+1} - u_{i-1}}{2h}\right)^2 + \frac{1}{2}\frac{u_N^2}{h^2}\right]$$
$$+\frac{h}{2}\left(\sum_1^N u_i^2\right) - \frac{h}{4}\left(\sum_1^N u_i^4\right) - ¸h\left[\frac{1}{2}\frac{w_1}{h}\frac{u_1}{h}\right.$$
$$\left.+\sum_1^N\left(\frac{w_{i+1} - w_{i-1}}{2h}\right)\left(\frac{u_{i+1} - u_{i-1}}{2h}\right) + \frac{1}{2}\frac{w_N}{h}\frac{u_N}{h}\right] \qquad (25)$$

where $u_1 = u_{N+1} = w_1 = w_{N+1} = 0$, $u = .u_1; u_2; \ldots; u_N/^t$, and $w = .w_1; w_2; \ldots; w_N/^t$ are, respectively, the column vectors of the nodes of the incremental lateral displacement and initial geometric imperfection.

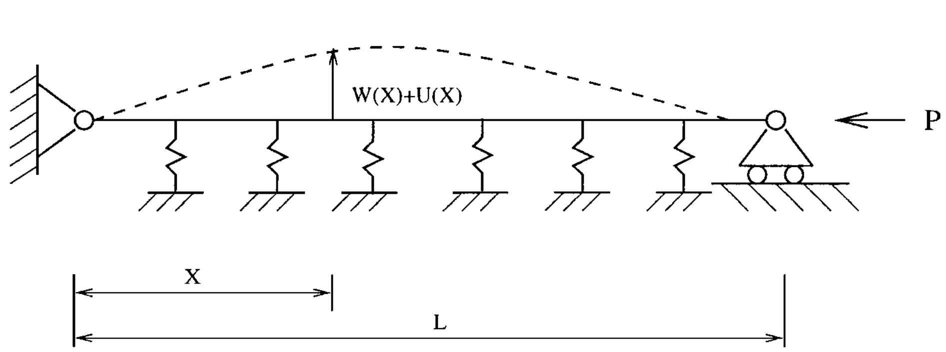

Fig. 3 Beam on a softening elastic foundation under axial step load of infinite duration.

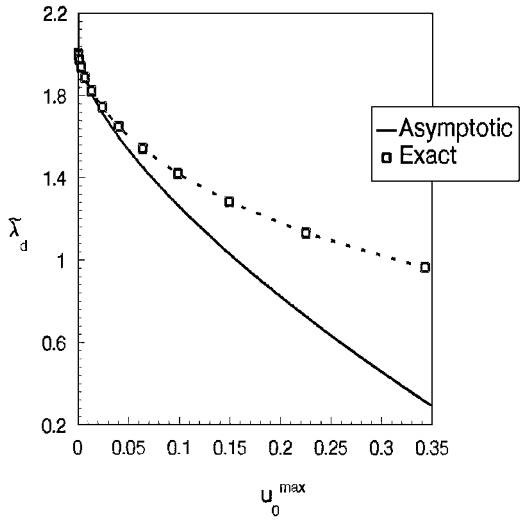

Fig. 4 Variation of lower-bound dynamic buckling load with maximum imperfection magnitude.

The components of the corresponding equilibrium equation $V_u = 0$ are

$$
\begin{aligned}
&u_3 - 4u_2 + \left(5 + h^4 - h^4 u_1^2\right) u_1 \\
&\quad - (h^2/4)\lambda(3u_1 - u_3 + 3w_1 - w_3) = 0 \\
&u_{i-2} - 4u_{i-1} - 4u_{i+1} + u_{i+2} + \left(6 + h^4 - h^4 u_i^2\right) u_i \\
&\quad - (h^2/4)\lambda(2u_i - u_{i-2} - u_{i+2} + 2w_i - w_{i-2} - w_{i+2}) \\
&= 0; \qquad 2 \le i \le N - 1 \\
&u_{N-2} - 4u_{N-1} + \left(5 + h^4 - h^4 u_N^2\right) u_N \\
&\quad + (h^2/4)\lambda(-3u_N + u_{N-2} - 3w_N + w_{N-2}) = 0
\end{aligned} \tag{26}
$$

In this finite-dimensional system, we define the inner product of u and v as follows:

$$(u, v) = h\left(\sum_{1}^{N} u_i v_i\right) \tag{27}$$

Let $N = 40$. The buckling load of the perfect beam is $\lambda_c = 2.002939$, and the corresponding buckling mode u_1 is normalized according to the norm $\|u_1\| = (u_1, u_1)^{1/2} = 1$ and its expression is omitted. In the computation, we consider the initial geometric imperfection, which has the shape of classical buckling mode u_1, and let the imperfection

$$w = \varepsilon u_1 \tag{28}$$

Based on the methods in Sec. III, extended system (8) is used to determine $v(\xi)$, $\lambda(\xi)$, $\varepsilon(\xi)$. The variation of the lower-bound dynamic buckling load $\tilde{\lambda}_d$ with the maximum magnitude u_0^{max} ($:= \varepsilon u_1^{max}$, where u_1^{max} is the largest component in the normalized buckling mode u_1) of the initial geometric imperfection is shown in Fig. 4. The comparison of the exact result, and the asymptotic solution [up to $O(\varepsilon)$] obtained by using the method described in Refs. 11 and 12, is best examined in Fig. 4.

Figure 4 indicates that the lower-bound dynamic buckling load decreases rapidly with the increase of the maximum initial geometric imperfection magnitude, and the asymptotic analysis is valid only for a small range of imperfection magnitude.

V. Conclusions

An extended system method for determining the lower-bound dynamic buckling loads of imperfection-sensitive systems under step load of infinite duration is presented. The method is based on the energy criterion for establishing the lower-bound dynamic buckling loads without solving the highly nonlinear initial-value problems. The solutions to the extended system form a locally unique smooth curve parameterized by ξ, the amplitude of projection of displacement u on the normalized buckling mode u_1 of the perfect system. Thus, with the extended system, standard methods can be used to compute the lower-bound dynamic buckling loads directly by using the continuation of ξ from $\xi = 0$. The implementation of Newton's method solving the extended system—a partitioning procedure—also is given. The two numerical examples show that one can get the lower-bound dynamic buckling load for sufficiently large imperfection magnitude and the algorithm has good convergence.

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